

SEMISIMPLE MAXIMAL QUOTIENT RINGS

BY

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Notation and Introduction. R denotes an arbitrary associative ring. A right R -module A over R will be denoted A_R . B_R is a large submodule of A_R (A_R is an essential extension of B_R), if B_R is a submodule of A_R having nonzero intersection with every nonzero submodule of A_R . A right ideal I of R is a large right ideal, if I_R is a large submodule of R_R .

Given A_R , $Z(A_R)$ is the singular submodule of A_R [9], which consists of all those elements of A whose annihilators in R are large right ideals.

Following Johnson [9], Q is a right quotient ring of R if Q is a ring with identity containing R as a subring (the identity of Q is the identity of R if R has one) and R_R is a large submodule of Q_R .

The quotient rings considered by Goldie in [6], [7] will be called classical quotient rings. Q is a classical right quotient of R if every regular element (nonzero divisor) of R is a unit in Q and every element of Q is of the form ab^{-1} , $a, b \in R$, b regular in R . In general, a ring R need not possess a classical right quotient ring.

Goldie [7], has given necessary and sufficient conditions that a ring possess a classical right quotient ring which is semisimple. Here semisimple means semisimple with minimum condition [8].

This paper is concerned with the question of characterizing those rings which have a semisimple maximal right quotient ring [4], [9], [10], [11] and in this case generalizing some simple well-known results about commutative integral domains, their quotient rings and modules over these domains. Johnson [9] has shown that R has a regular maximal right quotient ring Q if and only if $Z(R_R)=0$, where Q is a regular ring [13] if every finitely generated right (left) ideal of Q is generated by an idempotent. In this case Q_R is injective [3] as a right R -module, hence the injective hull of R [2].

A ring R has a semisimple maximal right quotient ring Q if and only if $Z(R_R)=0$ and $\dim R_R$ is finite, where a right R -module M is of finite dimension if every direct sum of submodules of M has only finitely many nonzero summands. This is the main result of §1. In addition another characterization is given for rings which possess a semisimple classical right quotient ring, namely, R has a semisimple classical right quotient ring if and only if $Z(R_R)=0$ and if I is a large right ideal of R , then there is an element $a \in I$ such that aR is a large right ideal of R .

If R has a semisimple classical right quotient ring Q , then it is known [3], that

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Q is the maximal right quotient ring of R . The converse is not valid, since there are rings with or without identity that have a semisimple maximal right quotient ring Q and Q is not the classical right quotient ring of R . Let Q be the ring of $n \times n$ matrices over a division ring Δ and R the set of upper triangular (strictly upper triangular) matrices of Q . It is easily verified that Q is a right quotient ring of R but Q is not a classical right quotient ring of R . Since Q is semisimple Q is the maximal right quotient ring of R .

It is also shown in §1 that if R has a semisimple maximal right quotient ring Q , then $Z(A/Z(A))=0$ for every right R -module A . This generalizes the analogous result if R is a commutative integral domain, since then $Z(A_R)$ is the torsion subgroup of A .

In §2 rings with identity are considered.

In this case the following generalizations of results known [1] when R is a commutative integral domain hold, thus extending some of the results of Gentile [5] also.

1. R has a semisimple maximal right quotient ring Q if and only if $A \rightarrow A \otimes_R Q$ has $Z(A_R)$ for its kernel for every unitary right R -module A .

If R has a semisimple maximal right quotient ring Q , then

2. ${}_R Q$ is flat [1] as a left R -module.

3. Every unitary left Q -module is flat as a left R -module.

4. If $Z(A_R)=0$, then $0 \rightarrow A \rightarrow A \otimes_R Q$ is the injective hull of A , a unitary R -module.

5. $\text{Tor}_1^R(A, Q/R) \cong Z(A_R)$ for every unitary right R -module A .

Another result with weaker hypothesis is valid.

Any direct sum of injective right R -modules, each with zero singular submodule is injective if $\dim R_R$ is finite.

The following generalizes a result of Matlis [12]. If H_R is an epimorphic image of an injective right R -module E_R and $Z(H/Z(H))=0$, then $Z(H)$ is a direct summand of H with complementary summand injective. The proof given here is simpler in that it does not appeal to any quotient ring of R as was done in Matlis [12, Theorem 1.1] when R is a commutative integral domain. Also Proposition 2.1, Proposition 2.2, and Proposition 2.4 of [12] can be generalized to a noncommutative ring R which has a semisimple maximal right quotient ring utilizing identical proofs.

1. Arbitrary rings.

DEFINITION 1.1. If M is a right R -module, then the set of all large submodules of M is denoted by $L(M_R)$.

It is useful to recall the following results, which are essentially in [9].

PROPOSITION 1.2.

1. If $A, B \in L(M_R)$, then $A+B \in L(M_R)$ and $A \cap B \in L(M_R)$.

2. If $A \in L(M_R)$ and B a submodule of M containing A , then $B \in L(M_R)$.

3. If A is a submodule of M , then there exists a submodule B of M maximal with respect to the property that $A \cap B = 0$ and consequently $A + B \in L(M_R)$.

4. If $f \in \text{Hom}_R(M, N)$ and $A \in L(N_R)$, then $f^{-1}(A) = \{x \in M \mid f(x) \in A\} \in L(M_R)$.

COROLLARY. If $A_1, \dots, A_n \in L(A_R)$ and $x_1, \dots, x_n \in A$, then $I = \{r \in R \mid x_i r \in A_i \text{ for all } i\} \in L(R_R)$.

Proof. $\{r \in R \mid x_i r \in A_i\} = I_i$ for each i is the counter-image of A_i by the right R homomorphism from R into A given by left multiplication by x_i , hence I is the intersection of finitely many large right ideals of R , so $I \in L(R_R)$.

Goldie [7] calls a right R -module M of finite dimension, $\dim M_R$ finite, if every direct sum of nonzero submodules of M has only a finite number of direct summands which are nonzero.

If M is a right R -module and $x_1, \dots, x_n \in M$, then $[x_1, \dots, x_n]$ will denote the submodule of M generated by $\{x_1, \dots, x_n\}$. For $x \in M$, $xR = \{xr \mid r \in R\}$. If R is a ring with identity and M a unitary R -module, then clearly $[x] = xR$. A module M_R will be called regular if for $0 \neq x \in M$, $xR \neq 0$. Clearly if R is a ring with identity every unitary R -module is regular. Also if R is arbitrary then M_R is regular if $Z(M_R) = 0$.

THEOREM 1.3. Let M_R be a right R -module, and consider the following conditions.

- (a) $\dim M_R$ is finite.
- (b) If $K \in L(M_R)$, there are $x_1, \dots, x_n \in K$ such that $[x_1, \dots, x_n] \in L(M_R)$.
- (c) If K is a submodule of M , then there are $x_1, \dots, x_n \in K$ such that $[x_1, \dots, x_n] \in L(K_R)$.
- (b*) If $K \in L(M_R)$, there are $x_1, \dots, x_n \in K$ such that $\sum x_i R \in L(M_R)$.
- (c*) If K is a submodule of M , then there are $x_1, \dots, x_n \in K$ such that $\sum x_i R \in L(K_R)$.

If M_R is any R -module then (a), (b), and (c) are equivalent. If M_R is a regular R -module then (a), (b*), and (c*) are equivalent, hence all the statements are equivalent.

Proof. Only the equivalence of (a), (b), and (c) will be shown as the equivalence of (a), (b*), and (c*) when M_R is regular has an analogous proof.

(a) implies (b). Let $K \in L(M_R)$ and suppose (b) is denied. Let $0 \neq x_1 \in K$, then $[x_1] \notin L(K_R)$, hence there is $0 \neq x_2 \in K$ such that $[x_1] \cap [x_2] = 0$. Suppose $x_1, \dots, x_n \in K$ such that $0 \neq [x_i]$ and the sum $\sum [x_i]$ is direct. Since $\sum [x_i] = [x_1, \dots, x_n]$, $\sum [x_i] \notin L(K_R)$, hence there is $0 \neq x_{n+1} \in K$ such that $[x_1, \dots, x_n] \cap [x_{n+1}] = 0$. Thus, there is an infinite sum $[x_1] \oplus \dots \oplus [x_n] \oplus \dots \subseteq K \subseteq M$ contradicting (a), so for some n $[x_1, \dots, x_n] \in L(K_R)$, hence $[x_1, \dots, x_n] \in L(M_R)$ since $K \in L(M_R)$.

(b) implies (c). Let K be a submodule of M_R . By Zorn's lemma there is a submodule L of M which is maximal with respect to the property that $K \cap L = 0$ and consequently $K + L \in L(M_R)$. By (b) there exist finitely many $a_1, \dots, a_n \in K + L$ such that $[a_1, \dots, a_n] \in L(M_R)$. Now $a_i = x_i + y_i$, $x_i \in K$, $y_i \in L$. The counter image of $[x_i, y_i]_{i=1}^n \in L(M_R)$ by the inclusion map $K \rightarrow K + L$ is $[x_1, \dots, x_n]$, hence is large in K , so (c) follows.

(c) implies (a). Given K a direct sum of nonzero submodules of M , then there are finitely many $x_1, \dots, x_n \in K$ such that $[x_1, \dots, x_n] \in L(K_R)$. Now $[x_1, \dots, x_n]$ is contained in the direct sum of finitely many of the submodules of K and $[x_1, \dots, x_n]$ thus has zero intersection with the others so the others are zero and (a) follows.

Similar proofs show the equivalence of (a), (b*), and (c*).

PROPOSITION 1.4. *For a unitary module A_R , the following statements are equivalent.*

- (a) A is semisimple (sum of simple submodules).
- (b) A is a direct sum of simple submodules.
- (c) Every submodule of A_R is a direct summand of A .
- (d) $L(A_R) = \{A_R\}$.

Proof. The equivalence of (a), (b), and (c) is well known, e.g., [8]. Clearly (c) implies (d). Conversely, if B is a submodule of A , then there is a submodule C of A such that $B \cap C = 0$ and $B + C \in L(A_R)$, by (d) $A = B \oplus C$ so (c) follows.

LEMMA 1.5. *If Q is a right quotient ring of R with $Z(R_R) = 0$ and A, B R -submodules of Q_R such that $A \cap B = 0$, then $AQ \cap BQ = 0$.*

Proof. If $x \in AQ \cap BQ$, then $x = \sum a_i q_i = \sum b_i p_i$, $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$, $p_1, \dots, p_n, q_1, \dots, q_n \in Q$. $I = \{r \in R \mid q_i r \in R, p_i r \in R \text{ for all } i\} \in L(R_R)$ and $xI = 0$ so $x \in Z(Q_R) = 0$, since $Z(R_R) = 0$.

COROLLARY. *If Q is a right quotient ring of R , $Z(R_R) = 0$ and $B \in L(Q_Q)$, then $B \cap L(R_R)$.*

Proof. If $B \cap J = 0$ for a right ideal J of R , then $BQ \cap JQ = 0$ by the lemma, hence $J = 0$ since $BQ = B$.

THEOREM 1.6. *Let $Z(R_R) = 0$, and Q the maximal right quotient ring of R , then the following statements are equivalent.*

- (a) $IQ = Q$ for every $I \in L(R_R)$.
- (b) For $I \in L(R_R)$ there are $a_1, \dots, a_n \in I$ such that $\sum a_i R \in L(R_R)$.
- (c) $\dim R_R$ is finite.
- (d) If I is a right ideal of R , then there are $a_1, \dots, a_n \in I$ such that $\sum a_i R \in L(I_R)$.
- (e) Q is a semisimple ring.

Proof. The equivalence of (b), (c), and (d) follows from Theorem 1.3 since $Z(R_R) = 0$.

(a) implies (b). If $I \in L(R_R)$, then $IQ = Q$, hence there are $a_1, \dots, a_n \in I$, $q_1, \dots, q_n \in Q$ such that $\sum a_i q_i = 1$. $J = \{r \in R \mid q_i r \in R \text{ for all } i\} \in L(R_R)$ and clearly $J \subseteq \sum a_i R$ so (b) follows.

(b) implies (e). If $B \in L(Q_Q)$, then $B \cap R \in L(R_R)$ by the corollary to Lemma 1.5. So $B \cap R$ has elements $a_1, \dots, a_n$ such that $I = \sum a_i R \in L(R_R)$. IQ is a finitely generated right ideal of Q . Since $Z(R_R) = 0$, Q is a regular ring, hence $IQ = eQ$, $e = e^2 \in Q$. However, $(1-e)I = 0$ so $1-e \in Z(Q_R) = 0$ so $IQ = Q$, but $IQ \subseteq B$ so $B = Q$, that is $L(Q_Q) = \{Q_Q\}$ so Q is a semisimple ring.

(e) implies (a). If $I \in L(R_R)$, then $IQ = eQ$ for some $e = e^2 \in Q$ since Q is semisimple. Since Q is the maximal right quotient ring of R and Q is semisimple, then Q is a regular ring so $Z(Q_R) = 0$ by [9]. Therefore, since $(1 - e)I = 0$, $(1 - e) \in Z(Q_R) = 0$, so $1 = e$ and $IQ = Q$.

Now the case of a semisimple classical right quotient ring of R will be considered.

THEOREM 1.7. *For a ring R , the following statements are equivalent.*

- (a) R has a semisimple classical right quotient ring.
- (b) $Z(R_R) = 0$ and for $I \in L(R_R)$ there is $a \in I$ such that $aR \in L(R_R)$.
- (c) R is a semiprime ring, $\dim R_R$ is finite and R satisfies the ascending chain condition on right annihilators.

Proof. The equivalence of (a) and (c) was shown by Goldie [7].

(a) implies (b)⁽²⁾. Let Q be a semisimple classical right quotient ring of R , then Q is the maximal right quotient ring of R . By Theorem 1.6 $Z(R_R) = 0$ and for $I \in L(R_R)$, there are $a_1, \dots, a_n \in I$, $q_1, \dots, q_n \in Q$ such that $\sum a_i q_i = 1$. Since $q_1 \in Q$, $q_1 = c_1 d_1^{-1}$, $c_1, d_1 \in R$, d_1 regular in R , hence $d_1 = \sum_{i>1} a_i q_i d_1 + a_1 c_1$. Since $q_2 d_1 \in Q$, $q_2 d_1 = c_2 d_2^{-1}$, $c_2, d_2 \in R$, d_2 regular in R , so $d_1 d_2 = a_1 c_1 d_2 + a_2 c_2 + \sum_{i>2} a_i q_i d_1 d_2$. Continuing in this fashion it follows that there exist regular elements $d_1, \dots, d_n \in R$ such that $d = d_1 \cdots d_n \in \sum a_i R \subseteq I$. If $dR \cap J = 0$ for a right ideal J of R , then $dRQ \cap JQ = 0$, but since $R \in L(R_R)$, $RQ = Q$ so $dRQ = Q$ since d is regular so $J = 0$, hence $dR \in L(R_R)$.

(b) implies (a). Let Q be the maximal right quotient ring of R and $q \in Q$, then $I = \{r \in R \mid qr \in R\} \in L(R_R)$. By (b) there is $a \in I$ such that $aR \in L(R_R)$. By Theorem 1.6 $Q = aRQ = aQ$ so a has right inverse. Since Q is semisimple and a has a right inverse a has a left inverse so a is a regular element of R and $q = ba^{-1}$. If $a \in R$ and a is regular, then the right annihilator of a in R is zero, hence in Q also. Since Q is semisimple a is regular in Q and (a) follows.

It is not valid in general that for a right R -module A_R , $Z(A/Z(A_R)) = 0$. Let R be a local ring with Jacobson radical $N \neq 0$ such that $N^2 = 0$. For instance $I/(p^2)$, I the ring of integers and p a prime. Since N is the unique maximal right ideal of R , $N \in L(R_R)$, hence it follows that $Z(R_R) = N$. Since $(R/N)N = 0$, $Z(R/N) = R/N \neq 0$.

THEOREM 1.8. *If $Z(R_R) = 0$ and $\dim R_R$ is finite, then $Z(A/Z(A)) = 0$ for every right R -module A_R .*

Proof. If $x + Z(A) \in Z(A/Z(A))$, then $I = \{r \in R \mid xr \in Z(A)\} \in L(R_R)$ by definition. By Theorem 1.6, Q , the maximal right quotient ring of R is semisimple so $IQ = Q$, hence there are $a_1, \dots, a_n \in I$, $q_1, \dots, q_n \in Q$ such that $\sum a_i q_i = 1$. For each i , $xa_i \in Z(A)$ so I_i , the annihilator of xa_i in R , is in $L(R_R)$. By the corollary to Proposition 1.2 $J = \{r \in R \mid q_i r \in I_i \text{ for each } i\} \in L(R_R)$. For $r \in J$, $xr = x(\sum a_i (q_i r)) = \sum xa_i (q_i r) = 0$, so $x \in Z(A)$ and the theorem follows.

⁽²⁾ Goldie [7] has shown this implication also.

This theorem raises the question of whether or not the condition $Z(A/Z(A))=0$ for every R -module is sufficient for R to possess a semisimple maximal ring of quotients.

2. Rings with identity. In this section R is a ring with unity 1, and all right R -modules are unitary.

LEMMA 2.1. *Let Q be a right quotient ring of R and A a right R -module, then if $a \otimes 1 = 0$ in $A \otimes_R Q$, there are finitely many $q_i \in Q$, $a_j \in A$, $a = a_1$, $\{r_{ij}\} \subseteq R$ such that*

$$\sum_i r_{ij} q_i = \delta_{1j} \quad (\text{Kronecker delta})$$

and $\sum_j a_j r_{ij} = 0$ for all i .

Proof. Let F be a free right R -module with basis $\{x_a : a \in A\}$, then the sequence

$$0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$$

of right R -modules is exact, where $(F \rightarrow A)(x_a) = a$, $K = \text{Ker}(F \rightarrow A)$. Tensoring over R with Q we have the exact sequence

$$K \otimes Q \rightarrow F \otimes Q \rightarrow A \otimes Q \rightarrow 0.$$

If $a \otimes 1 = 0$ in $A \otimes Q$, then $x_a \otimes 1$ is the image of an element from $K \otimes Q$. $x_a \otimes 1 = \sum_i k_i \otimes q_i$. Since $k_i \in K \subseteq F$, $k_i = \sum_j x_{a_j} \lambda_{ij}$ for each i , a finite sum. Now $x_a \otimes 1 = \sum_i (\sum_j x_{a_j} \lambda_{ij}) \otimes q_i = \sum_j x_{a_j} \otimes (\sum_i \lambda_{ij} q_i)$. Since representation in $F \otimes Q$ is unique with respect to basis elements $x_a = x_{a_j}$ for some j say $j=1$, hence $\sum_i \lambda_{ij} q_i = \delta_{1j}$ and from $k_i = \sum_j x_{a_j} \lambda_{ij}$, $0 = \sum_j a_j \lambda_{ij}$ for all i , and the lemma follows.

PROPOSITION 2.2. *If Q is a right ring of quotients of R , A a right R -module, then the kernel of the map $A \otimes R \rightarrow A \otimes Q$ is contained in $Z(A_R)$.*

Proof. If $a \otimes 1 = 0$ in $A \otimes Q$, then by Lemma 1.3 there exist finitely many $\{q_i\} \subseteq Q$, $\{a_j\} \subseteq A$, $a_1 = a$, $\{\lambda_{ij}\} \subseteq R$ such that

$$\sum_i \lambda_{ij} q_i = \delta_{1j}, \quad \sum_j a_j \lambda_{ij} = 0 \quad \text{for all } i.$$

Let $I = \{r \in R \mid q_i r \in R \text{ for each } i\}$, then I is a large right ideal of R by the corollary to Proposition 1.2 and for $\lambda \in I$

$$\begin{aligned} 0 &= \sum_i \left(\sum_j a_j \lambda_{ij} \right) (q_i \lambda) = \sum_{i,j} a_j \lambda_{ij} (q_i \lambda) \\ &= \sum_j a_j \left(\sum_i \lambda_{ij} (q_i \lambda) \right) = \sum_j a_j (\delta_{1j} \lambda) = a_1 \lambda = a \lambda, \end{aligned}$$

hence a is annihilated by I , so $a \in Z(A_R)$.

THEOREM 2.3. *If Q is the maximal right quotient ring of R , then (a), (b), (c), (d), (e) of Theorem 1.6 are all equivalent to (f) $\text{Ker}(A \otimes_R R \rightarrow A \otimes_R Q) = Z(A_R)$ for every right R -module A .*

Proof. (a) implies (f). By Proposition 2.2, $\text{Ker}(A \otimes R \rightarrow A \otimes Q) \subseteq Z(A_R)$. If $a \in Z(A_R)$ then I , the annihilator of a in R , is in $L(R_R)$ so $IQ = Q$, hence there are $a_1, \dots, a_n \in I$, $q_1, \dots, q_n \in Q$ such that $\sum a_i q_i = 1$. In $A \otimes Q$, $a \otimes 1 = a \otimes (\sum a_i q_i) = \sum aa_i \otimes q_i = 0$ so (f) follows.

(f) implies (a). If $I \in L(R_R)$, then from the exact sequence $I \otimes Q \rightarrow R \otimes Q \rightarrow R/I \otimes Q \rightarrow 0$ we have that $R/I \otimes Q$ is isomorphic to Q/IQ . Since $I \in L(R_R)$, $Z(R/I) = R/I$, so $R/I \otimes R \rightarrow R/I \otimes Q$ is the zero map hence $\bar{1} \otimes 1 = 0$ in $R/I \otimes Q$. However, $R/I \otimes Q \cong Q/IQ$ is a right Q -module generated by $\bar{1} \otimes 1$ so $Q/IQ = 0$, hence $I \otimes Q \rightarrow Q$ is onto. It is now clear that (a) follows since the image of $I \otimes Q \rightarrow Q$ in Q is IQ .

An immediate consequence of the notion of singular submodule is

PROPOSITION 2.4. *If $E_R = \bigoplus_i E_i$, E_i right R -modules then $Z(E) = \bigoplus_i Z(E_i)$.*

If R is a commutative integral domain, then any direct sum of torsion free injective R -modules is injective, since it is torsion-free and divisible, hence injective, [1, Proposition VII.1.3]. A generalization holds.

THEOREM 2.5. *If $\dim R_R$ is finite and E_R is the direct sum of injectives which have zero singular submodule, then E is injective.*

Proof. By Proposition 2.3 $Z(E) = 0$. It is sufficient to show that every R -homomorphism from a large right ideal of R into E can be extended to R . Let $f \in \text{Hom}(I_R, E_R)$, $I_R \in L(R_R)$. By Theorem 1.3, there exist finitely many $a_1, \dots, a_n \in I$ such that $J = \sum a_i R \in L(R_R)$. Let f' be the restriction of f to J . Since J is finitely generated, $f'(J)$ is contained in a finite direct sum of injectives, hence f' has an extension $f^* \in \text{Hom}(R_R, E_R)$. The assertion is that f^* is an extension of f . Let $x \in I$, then $K = \{r \in R \mid xr \in J\} \in L(R_R)$. Now for $r \in K$, $(f(x) - f^*(x))r = f(xr) - f^*(xr) = f'(xr) - f'(xr) = 0$ so $f(x) - f^*(x) \in Z(E) = 0$, hence $f^*(x) = f(x)$.

It is known [3], that if $Z(R_R) = 0$, then Q the maximal right ring of R is injective as a right R -module.

THEOREM 2.6. *If Q is the maximal right quotient ring of R , $Z(R_R) = 0$, $\dim R_R$ finite and A_R a right R -module such that $Z(A_R) = 0$, then the map $0 \rightarrow A \rightarrow A \otimes_R Q$ is an injective hull of A as a right R -module.*

Proof. The map is a monomorphism by Theorem 2.3.

Now $A \otimes_R Q$ is a right Q module, hence semisimple since Q is, so $A \otimes_R Q$ is a direct sum of direct summands of Q . Since $Z(Q_R) = 0$, $Z(A \otimes_R Q) = 0$ regarding $A \otimes Q$ as a right R -module and by Theorem 2.5 $A \otimes_R Q$ is injective as a right R -module.

If $0 \neq x = \sum a_i \otimes q_i \in A \otimes Q$, then $I = \{r \in R \mid q_i r \in R\} \in L(R_R)$. Now $0 \neq xI$ since $Z(A \otimes Q) = 0$ and $xI \subseteq \text{Im}(A \rightarrow A \otimes Q)$ so $0 \rightarrow A \rightarrow A \otimes Q$ is an essential monomorphism, i.e., $\text{Im}(A \rightarrow A \otimes Q)$ is a large right R submodule of $A \otimes Q$ and the theorem follows.

THEOREM 2.7. *If Q is a semisimple maximal right quotient ring of R , then Q is flat as a left R -module.*

Proof. It is sufficient to show that $I \otimes_R Q \rightarrow R \otimes_R Q$ is a monomorphism for every right ideal of R . As before $I \otimes_R Q$ is a right Q -module, then $Z(I \otimes Q) = 0$ regarding $I \otimes Q$ as a right R -module. If $x = \sum a_i \otimes q_i \in \text{Ker}(I \otimes Q \rightarrow R \otimes Q)$, then $I = \{r \in R \mid q_i r \in R\} \in L(R_R)$ and $\sum a_i q_i = 0$ in Q . Clearly $xI = 0$, so $x \in Z(I \otimes Q) = 0$, so $x = 0$, hence $I \otimes Q \rightarrow R \otimes Q$ is a monomorphism.

COROLLARY. *If Q is the maximal right quotient ring of R , $Z(R_R) = 0$, $\dim R_R$ finite, then $\text{Tor}_1^R(A, Q/R) \cong Z(A_R)$ for every right R -module A_R .*

Proof. It follows from the exact sequence $\text{Tor}_1^R(A, Q) \rightarrow \text{Tor}_1^R(A, Q/R) \rightarrow A \otimes_R R \rightarrow A \otimes_R Q$ since by the theorem $\text{Tor}_1^R(A, Q) = 0$ and $\text{Ker}(A \otimes_R R \rightarrow A \otimes_R Q) \cong Z(A_R)$ by Theorem 2.3.

COROLLARY. *If Q is a semisimple maximal right quotient of R , then every left Q -module is flat as a left R -module.*

Proof. Every left Q -module is a direct sum of direct summands of Q , hence is flat as a left R -module since Tor_n^R commutes with direct sum s .

Matlis [12, Theorem 1.1] has shown that if R is a commutative integral domain and H an R -module, then the torsion submodule of H is a direct summand of H , if H is an epimorphic image of an injective R -module. This result is generalized and the proof does not appeal to the quotient ring of R .

First, the notion of a closed submodule of a module will be considered and some consequences. Johnson and Wong [11] considered the notion of a closed submodule.

DEFINITION 2.8. A submodule B of a module A is closed if B has no essential extension in A ; i.e., C a submodule of A such that B is a large submodule of C implies $B = C$.

REMARK. If E is an injective R -module and A a submodule of E , then A is closed if and only if A is a direct summand of E . This follows from the fact that every submodule of E is a large submodule of its injective hull in E , which is a direct summand of E .

LEMMA 2.9. *If $f \in \text{Hom}(M, A)$, B a submodule of A such that $Z(A/B) = 0$, then $f^{-1}(B)$ is a closed submodule of M .*

Proof. Let D be a submodule of M containing $f^{-1}(B)$ as a large submodule. If $d \in D$, then $I = \{r \in R \mid dr \in f^{-1}(B)\} \in L(R_R)$. Now $f(d)I = f(dI) \subseteq B$, hence $f(d) + B = [f(d)]^- \in Z(A/B) = 0$ so $f(d) \in B$ and $d \in f^{-1}(B)$ and the lemma follows.

THEOREM 2.10. *If $E \xrightarrow{f} H \rightarrow 0$ is an exact sequence of right R -module, E injective, $Z(H/Z(H)) = 0$, then $H = Z(H) \oplus F$ and F is injective.*

Proof. Since $Z(H/Z(H)) = 0$, by Lemma 2.9 $f^{-1}(Z(H))$ is closed in E , hence a direct summand of $E = f^{-1}(Z(H)) \oplus G$. Clearly $H = Z(H) \oplus f(G)$ and $f(G) \cong G$.

COROLLARY. *If R is a ring such that $Z(R_R)=0$, $\dim R_R$ finite, then every epimorphic image of an injective R -module has its singular submodule as a direct summand.*

Proof. An immediate consequence of the theorem and Theorem 1.8.

It is interesting to note that some of the propositions of [12] admit generalizations to noncommutative rings and their maximal right quotient rings, where torsion submodule is replaced with singular submodule and Q the maximal right quotient ring of R .

If in addition to the hypothesis of [12, Proposition 2.1] we assume $Z(R_R)=0$, $\dim R_R$ finite, then Proposition 2.1 is valid with the same proofs using the fact that $Z(Q_R)=0$ and a direct sum of copies of Q is injective by Theorem 2.5. Similarly a generalization of [12, Proposition 2.2] is valid in view of the corollary to Theorem 2.11, as well as [12, Corollary 2.3] of [12, Proposition 2.2].

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